International Journal of Theoretical Physics, Vol. 36, No. 3, 1997

Lax Representations and Zero-Curvature Representations by the Kronecker Product

Wen-Xiu Ma[1,2] and Fu-Kui Guo[3]

Received July 23, 1996

It is shown that the Kronecker product can be applied to construct not only new Lax representations, but also new zero-curvature representations of integrable models. A characteristic difference between continuous and discrete zero-curvature equations is pointed out.

Lax and zero-curvature representations play important roles in studying nonlinear integrable models in theoretical physics. It is based on such representations that the inverse scattering transform is developed (see, e.g., Ablowitz and Clarkson, 1991). They may also provide a great deal of information on, e.g., integrals of motion, master symmetries, and Hamiltonian formulations. There exist many integrable models that possess a Lax or zero-curvature representation (Faddeev and Takhtajan, 1987; Das, 1989). Typical examples are the Toda lattice (Flaschka, 1974) and AKNS systems (Ablowitz *et al.*, 1974), including the KdV equation and the nonlinear Schrödinger equation.

In this paper we present new Lax representations and new zero-curvature representations by using the Kronecker product of matrices, motivated by recent work of Steeb and Heng (1996). The Kronecker product itself has nice mathematical properties and important applications in many fields of physics, for example, statistical physics, quantum groups, etc. (Steeb, 1991). Our result for the zero-curvature representation also provides us with a characteristic difference between continuous and discrete zero-curvature equations.

[1] Institute of Mathematics, Fudan University, Shanghai 200433, China.
[2] Mathematik und Informatik, Universität-GH Paderborn, D-33098 Paderborn, Germany.
[3] Department of Applied Mathematics, Shandong Mining Institute, Tai'an 271019, China.

0020-7748/97/0300-0697$12.50/0 © 1997 Plenum Publishing Corporation

Let I_M denote the unit matrix of order M, $M \in \mathbb{Z}$. For two matrices $A = (a_{ij})_{pq}$, $B = (b_{kl})_{rs}$, the Kronecker product $A \otimes B$ is defined by (Steeb, 1991)

$$A \otimes B = (a_{ij}B)_{(pr)\times(qs)} \tag{1}$$

or equivalently by (Hoppe, 1992)

$$(A \otimes B)_{ij,kl} = a_{ik}b_{jl} \tag{2}$$

Evidently we have a basic relation on the Kronecker product (Steeb, 1991; Hoppe, 1992)

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{3}$$

provided that the matrices AC and BD make sense. This relation will be used to show a new structure of Lax and zero-curvature representations of integrable models.

Theorem 1 (Lax representation). Assume that an integrable model (continuous or discrete) has two Lax representations

$$L_{1t} = [A_1, L_1], \qquad L_{2t} = [A_2, L_2] \tag{4}$$

where L_1, A_1 and L_2, A_2 are $M \times M$ and $N \times N$ matrices, respectively. Define

$$L_3 = \alpha_1 L_1 \otimes L_2 + \alpha_2(L_1 \otimes I_N + I_M \otimes L_2), \qquad A_3 = A_1 \otimes I_N + I_M \otimes A_2 \tag{5}$$

where α_1, α_2 are arbitrary constants. Then the same integrable model has another Lax representation, $L_{3t} = [A_3, L_3]$.

Proof. First of all, we have

$$L_{3t} = \alpha_1(L_{1t} \otimes L_2 + L_1 \otimes L_{2t}) + \alpha_2(L_{1t} \otimes I_N + I_M \otimes L_{2t}) \tag{6}$$

On the other hand, using (3), we can calculate that

$$\begin{aligned}[A_3, L_3] &= \alpha_1([A_1, L_1] \otimes L_2 + L_1 \otimes [A_2, L_2]) \\ &\quad + \alpha_2([A_1, L_1] \otimes I_N + I_M \otimes [A_2, L_2])\end{aligned}$$

Now we easily find that the equalities defined by (4) imply $L_{3t} = [A_3, L_3]$. ■

When $\alpha_2 = 0$ the result obtained is exactly that in Steeb and Heng (1996). When $\alpha_1 = 0$ we get a new Lax representation for a given integrable model, starting from two known Lax representations. Integrals of motion may also be generated from the new Lax representation, because we have

$$\begin{aligned}F_{ij} &= \mathrm{tr}(\alpha_1 L_1^i \otimes L_2^j + \alpha_2(L_1^i \otimes I_N + I_M \otimes L_2^j)) \\ &= \alpha_1 \, \mathrm{tr}(L_1^i) \, \mathrm{tr}(L_2^j) + \alpha_2(N \, \mathrm{tr}(L_1^i) + M \, \mathrm{tr}(L_2^j))\end{aligned} \tag{7}$$

where we have used $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$ (Steeb, 1991) and $(L_1^i)_t = [A_1, L_1^i]$, $(L_2^i)_t = [A_2, L_2^i]$.

Theorem 2 (Continuous zero-curvature representation). Assume that a continuous integrable model has two continuous zero-curvature representations

$$U_{1t} - V_{1x} + [U_1, V_1] = 0, \qquad U_{2t} - V_{2x} + [U_2, V_2] = 0 \tag{8}$$

where U_1, V_1 and U_2, V_2 are $M \times M$ and $N \times N$ matrices, respectively. Define

$$U_3 = U_1 \otimes I_N + I_M \otimes U_2, \qquad V_3 = V_1 \otimes I_N + I_M \otimes V_2 \tag{9}$$

Then the same integrable model has another continuous zero-curvature representation

$$U_{3t} - V_{3x} + [U_3, V_3] = 0 \tag{10}$$

Proof. The proof is also a direct computation. We first have

$$U_{3t} = U_{1t} \otimes I_N + I_M \otimes U_{2t}$$
$$U_{3x} = U_{1x} \otimes I_N + I_M \otimes U_{2x}$$

Second, using (3), we can obtain that

$$[U_3, V_3] = [U_1, V_1] \otimes I_N + I_M \otimes [U_2, V_2] \tag{11}$$

Therefore we see that (10) is true once two equalities defined by (8) hold. ■

We remark that when we choose

$$U_3 = U_1 \otimes U_2$$

the third zero-curvature representation (10) is not guaranteed to be true. An example will be displayed below.

Theorem 3 (Discrete zero-curvature representation). Assume that a discrete integrable model has two discrete zero-curvature representations

$$U_{1t} = (EV_1)U_1 - U_1V_1, \qquad U_{2t} = (EV_2)U_2 - U_2V_2 \tag{12}$$

where E is the shift operator, U_1, V_1 are $M \times M$ matrices, and U_2, V_2 are $N \times N$ matrices. Define

$$U_3 = U_1 \otimes U_2, \qquad V_3 = V_1 \otimes I_N + I_M \otimes V_2 \tag{13}$$

Then the same integrable model has another discrete zero-curvature representation

$$U_{3t} = (EV_3)U_3 - U_3V_3 \tag{14}$$

Proof. Similarly, we first have

$$U_{3t} = U_{1t} \otimes U_2 + U_1 \otimes U_{2t} \tag{15}$$

On the other hand, we may calculate that

$$\begin{aligned}(EV_3)U_3 - U_3V_3 &= ((EV_1) \otimes I_N + I_M \otimes (EV_2))(U_1 \otimes U_2) \\ &\quad - (U_1 \otimes U_2)(V_1 \otimes I_N + I_M \otimes V_2) \\ &= ((EV_1)U_1) \otimes U_2 + U_1 \otimes ((EV_2)U_2) \\ &\quad - (U_1V_1) \otimes U_2 - U_1 \otimes (U_2V_2) \\ &= ((EV_1)U_1 - U_1V_1) \otimes U_2 + U_1 \otimes ((EV_2)U_2 - U_2V_2)\end{aligned}$$

In the second equality above, we used the basic relation (3). Hence we find that (14) holds if we have (12). ■

We remark that when we choose

$$U_3 = U_1 \otimes I_M + I_N \otimes U_2$$

the third discrete zero-curvature representation (14) is not guaranteed to be true. An example will also be given below. This is opposite to the result in the continuous case. It shows us a characteristic difference between continuous and discrete zero-curvature equations.

In what follows we show some concrete examples to illustrate the use of the above Kronecker product technique. Actually once we have a Lax representation or a zero-curvature representation, we can obtain a new representation after choosing two required representations to be this known one. Another new representation may be constructed by use of this new representation and the process may be infinitely extended. This also tells us that there exist infinitely many Lax representations or zero-curvature representations once there exists one representation for a given integrable model. The construction procedure will be shown in the following examples and can be easily generalized to other integrable models; see, for example, Calogero and Nucci (1991), Drinfel'd and Sokolov (1984), Ma (1993b), Ragnisco and Santini (1990), and Tu (1990).

Example 1. We consider periodic Toda lattice (Flaschka, 1974)

$$a_{it} = a_i(b_{i+1} - b_i), \qquad b_{it} = 2(a_i^2 - a_{i-1}^2), \qquad a_{i+N} = a_i, \qquad b_{i+N} = b_i \tag{16}$$

which is a Hamiltonian system with Hamiltonian

$$H(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} e^{q_i - q_{i+1}}$$

under the Flaschka transformation

$$a_i = \tfrac{1}{2}e^{q_i - q_{i+1}}, \qquad b_i = -\tfrac{1}{2}p_i$$

The Toda lattice (16) has a Lax representation with

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & a_N \\ a_1 & b_2 & a_2 & \cdots & \cdots & 0 \\ 0 & a_2 & b_3 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ a_N & \cdots & \cdots & \cdots & a_{N-1} & b_N \end{pmatrix} \tag{17}$$

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots & -a_N \\ -a_1 & 0 & a_2 & \cdots & \cdots & 0 \\ 0 & -a_2 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ a_N & \cdots & \cdots & \cdots & -a_{N-1} & 0 \end{pmatrix} \tag{18}$$

Through Theorem 1, we obtain a new Lax representation with

$$L_{\text{new}} = \alpha_1 L \otimes L + \alpha_2(L \otimes I_N + I_N \otimes L), \qquad A_{\text{new}} = A \otimes I_N + I_N \otimes A \tag{19}$$

Here α_1, α_2 are two arbitrary constants and thus the Toda lattice (16) has many different Lax representations. By (7), new integrals of motion may be generated, which are all functions of $F_i = \text{tr}(L^i)$.

Example 2. The nonlinear Schrödinger model (Ablowitz *et al.*, 1974; Ma and Strampp, 1994)

$$\begin{cases} p_t = -\tfrac{1}{2}q_{xx} + p^2 q \\ q_t = \tfrac{1}{2}p_{xx} - pq^2 \end{cases} \tag{20}$$

has a continuous zero-curvature representation with

$$U = \begin{pmatrix} -\lambda & p \\ q & \lambda \end{pmatrix}, \qquad V = \begin{pmatrix} -\lambda^2 + \tfrac{1}{2}pq & \lambda p - \tfrac{1}{2}p_x \\ \lambda q + \tfrac{1}{2}q_x & \lambda^2 - \tfrac{1}{2}pq \end{pmatrix} \tag{21}$$

This model has infinitely many symmetries and integrals of motion. According to Theorem 2, we obtain new continuous zero-curvature representations with

$$U_{\text{new}} = U \otimes I_2 + I_2 \otimes U$$
$$= \begin{pmatrix} -2\lambda & p & p & 0 \\ q & 0 & 0 & p \\ q & 0 & 0 & p \\ 0 & q & q & 2\lambda \end{pmatrix} \tag{22}$$

$$V_{\text{new}} = V \otimes I_2 + I_2 \otimes V$$
$$= \begin{pmatrix} -2\lambda^2 + pq & \lambda p - \frac{1}{2}p_x & \lambda p - \frac{1}{2}p_x & 0 \\ \lambda q + \frac{1}{2}q_x & 0 & 0 & \lambda p - \frac{1}{2}p_x \\ \lambda q + \frac{1}{2}q_x & 0 & 0 & \lambda p - \frac{1}{2}p_x \\ 0 & \lambda q + \frac{1}{2}q_x & \lambda q + \frac{1}{2}q_x & 2\lambda^2 - pq \end{pmatrix} \tag{23}$$

or with

$$U_{\text{new}} = U \otimes I_4 + I_2 \otimes \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \qquad V_{\text{new}} = V \otimes I_4 + I_2 \otimes \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \tag{24}$$

The spectral operator defined by (22) is similar to one appearing in Khasilev (1992) and we may also discuss its binary nonlinearization [for the cases of 2×2 and 3×3 matrices see Ma and Strampp (1994) and Ma *et al.* (1996)]. However, the nonlinear Schrödinger model (20) does not have the continuous zero-curvature representation with

$$U_{\text{new}} = U \otimes U, \qquad V_{\text{new}} = V \otimes I_2 + I_2 \otimes V \tag{25}$$

Example 3. We consider the Volterra lattice (Fuchssteiner and Ma, 1996)

$$u_t = u(u^{(-1)} - u^{(1)}), \qquad u^{(m)} = E^m u \tag{26}$$

More examples of lattices may be found in Steeb (1991), Ragnisco and Santini (1990), and Tu (1990), for example, the one-dimensional isotropic Heisenberg model. The lattice (26) has a discrete zero-curvature representation with

$$U = \begin{pmatrix} 1 & u \\ \lambda^{-1} & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} \frac{1}{2}\lambda - u & \lambda u \\ 1 & -\frac{1}{2}\lambda - u^{(-1)} \end{pmatrix} \tag{27}$$

By Theorem 3 we obtain new discrete zero-curvature representations with

$$U_{\text{new}} = U \otimes \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \qquad V_{\text{new}} = V \otimes I_4 + I_2 \otimes \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \tag{28}$$

or with

$$U_{\text{new}} = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \otimes \begin{pmatrix} U & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U \end{pmatrix}$$

$$V_{\text{new}} = \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \otimes I_6 + I_4 \otimes \begin{pmatrix} V & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & V \end{pmatrix} \tag{29}$$

The latter is made up of two 24×24 matrices. To find directly these two matrices requires much complicated calculation. Note that we do not have the discrete zero-curvature representation with

$$U_{\text{new}} = U \otimes I_2 + I_2 \otimes U, \qquad V_{\text{new}} = V \otimes I_2 + I_2 \otimes V \tag{30}$$

for the Volterra lattice (26). This is not strange and shows the difference between the two kinds of zero-curvature representations.

Finally we present an open problem. We denote the Gateaux derivative $K'[S]$ by $K'[S] = (\partial/\partial\epsilon)K'(u + \epsilon S)|_{\epsilon=0}$. We have already established the following result (Ma, 1992, 1993a; Fuchssteiner and Ma, 1996): If $u_t = K(u)$, $u_t = S(u)$ have Lax representations

$$L_t = [A_1, L], \qquad L_t = [A_2, L]$$

or zero-curvature representations

$$U_t - V_{1x} + [U, V_1] = 0 \qquad [\text{or } U_t = (EV_1)U - UV_1]$$

$$U_t - V_{2x} + [U, V_2] = 0 \qquad [\text{or } U_t = (EV_2)U - UV_2]$$

respectively, then the product model $u_t = [K, S] := K'[S] - S'[K]$ has Lax representation

$$L_t = [A_3, L], \qquad A_3 = A_1'[S] - A_2'[K] + [A_1, A_2]$$

or zero-curvature representation

$$U_t - V_{3x} + [U, V_3] = 0 \quad [\text{or } U_t = (EV_3)U - UV_3],$$
$$V_3 = V_1'[S] - V_2'[K] + [V_1, V_2]$$

Therefore $u_t = [K, S]$ may have a Lax representation with the spectral operator and the Lax operator determined by the Kronecker product. For example, in the case of the Lax representation we have

$$L_{\text{new}} = \alpha_1 L \otimes L + \alpha_2(L \otimes I_M + I_M \otimes L)$$

$$A_{\text{new}} = (A_1'[S] - A_2'[K] + [A_1, A_2]) \otimes I_M + I_M \otimes (A_1'[S] - A_2'[K] + [A_1, A_2])$$

where M is the order of the matrix L. Product models may be applied to construct symmetries of nonlinear models and thus they are important. Let us now suppose that two models $u_t = K(u)$, $u_t = S(u)$ have two completely different Lax representations

$$L_{1t} = [A_1, L_1], \qquad L_{2t} = [A_2, L_2]$$

or two completely different zero-curvature representations

$$U_{1t} - V_{1x} + [U_1, V_1] = 0 \qquad [\text{or } U_t = (EV_1)U - UV_1]$$

$$U_{2t} - V_{2x} + [U_2, V_2] = 0 \qquad [\text{or } U_t = (EV_2)U - UV_2]$$

Here L_1 and L_2 or U_1 and U_2 are not equal, and sometimes they may have different matrix orders. The problem is to determine the corresponding representation for the product model $u_t = [K, S]$. It seems to us that the required spectral operator matrix L_{new} or U_{new} should be represented by some Kronecker product involving L_1, L_2 or U_1, U_2 and K, S.

ACKNOWLEDGMENTS

This work was supported by the Alexander von Humboldt Foundation and the National Natural Science Foundation of China.

REFERENCES

Ablowitz, M. J., and Clarkson, P. A. (1991). *Solitons, Nonlinear Evolution Equations and Inverse Scattering,* Cambridge University Press, Cambridge.

Ablowitz, M. J., Kaup, D. J., Newell, A. C., and Segur, H. (1974). *Studies in Applied Mathematics,* **53**, 249.

Calogero, F., and Nucci, M. C. (1991). *Journal of Mathematical Physics,* **32**, 72.

Das, A. (1989). *Integrable Models,* World Scientific, Singapore.

Drinfel'd, V. G., and Sokolov, V. V. (1984). *Soviet Journal of Mathematics,* **24**, 81.

Faddeev, L. D., and Takhtajan, L. A. (1987). *Hamiltonian Methods in the Theory of Solitons,* Springer-Verlag, Berlin.

Flaschka, H. (1974). *Physical Review B,* **9**, 1924.

Fuchssteiner, B., and Ma, W. X. (1996). An approach to master symmetries of Lattice equations, in *Proceedings of Symmetries and Integrability of Differential Equations, Canterbury, UK,* to appear.

Hoppe, J. (1992). *Lectures on Integrable Systems,* Springer-Verlag, Berlin.

Khasilev, V. Ya. (1992). *JETP Letters,* **56**, 194.

Ma, W. X. (1992). *Journal of Mathematical Physics,* **33**, 2464.

Ma, W. X. (1993a). *Journal of Physics A: Mathematical and General,* **26**, 2573.

Ma, W. X. (1993b). *Journal of Physics A: Mathematical and General,* **26**, L1169.

Ma, W. X., Fuchssteiner, B., and Oevel, W. (1996). *Physica A,* **233**, 331.

Ma, W. X., and Strampp, W. (1994). *Physics Letters A,* **185**, 277.

Ragnisco, O., and Santini, P. M. (1990). *Inverse Problems,* **6**, 441.

Steeb, W.-H. (1991). *Kronecker Product and Applications,* BI-Wissenschaftsverlag, Mannheim.

Steeb, W.-H., and Heng, L. C. (1996). *International Journal of Theoretical Physics,* **35**, 475.

Tu, G. Z. (1990). *Journal of Physics A: Mathematical and General,* **23**, 3903.